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# EQUATIONS OF MOTION OF GRANULAR MEDIA


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Acoustical studies of granular media used in petroleum and gas collectors have recently uncovered a number of unusual phenomena. Thus in highly porous bodies with empty or gas-saturated voids, the ratio of the velocities of S and P waves is often inexplicably large ($V_S/V_P > 1/\sqrt{2}$), which corresponds formally to negative values of the Poisson coefficient. According to data of [1] and other studies the value of $V_S/V_P$ sometimes exceeds 0.75, i.e., the Poisson coefficient is less than −0.3. Moreover, wave velocities measured by various authors in the same specimens differ among themselves greatly (up to 10–15%), although in "good" test specimens (metallic ones, for example), they practically coincide. The experimental data indicate the insufficient development of physical theories of weak wave propagation in granular media such as hydrocarbon collectors.

Granular media possess two important unique features. First the linear dimensions of the grains allow introduction of a new dimensionless characteristic differing from the porosity f, which describes the pore space, namely $\eta = \sigma_0 r_0/3$, $\sigma_0$, the specific surface of the porous body, where $r_0$ is the mean grain radius. It has been proven by integral geometry that $0 \leqslant \eta \leqslant 1 - f$. Second, the presence of contacts between grains and sections of grains free from stress leads to a complex stressed state in each grain taken individually, so that aside from the mean (large scale) field, which changes markedly at distances of the order of a wavelength, a fluctuation field develops, which varies significantly at distance of the order of the individual grain size. Development of the fluctuation field leads to scattering of the energy contained in waves which are no longer purely P and S waves at each individual point, but only on the average. This implies that P and S waves are formed only by the average (large scale) stress and deformation fields, while fluctuations insure scattering of waves and a decrease in the amplitude of the mean field. In constructing a model of a continuous medium equivalent to a granular skeleton, the two features of the microinhomogeneous medium mentioned above must be considered. It is insufficient merely to require free equivalency of the media in the sense that the ratios of stress to deformation for the skeleton and the continuous models coincide. The presence of scattering and attenuation of the large scale field must lead to some wave "absorption" mechanism, produced by the scattering.

The above considerations demand a precise solution of the problem of elastic equilibrium for an individual grain, which in principle can be given by the ratio of stress to deformation at the center of the grain (i.e., the mean values of $\lambda$ and $\mu$ in the structure) and the fraction of energy $\alpha$ contained in the fluctuation field referred to the mean field. These constants, which depend on the geometry of the pore space and material of the skeleton, allow transition to construction of an equation of motion of some set of particles with known mean values of the Lamé coefficients and known fraction of the energy scattered. It can be expected that the presence of isotropic scattering is equivalent to introduction of additional randomly oriented sources which collect the energy of the large scale field, attenuating the latter. The goal of the present study is to derive equations of motion (and equilibrium) for the mean field, since it is only this mean field which is recorded by any device utilizing


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a sufficiently large set of particles, the mean values of the fluctuation stresses and deformations being equal to zero. Thus, with regard to force, the fluctuation field is equivalent to zero, yet with respect to energy its influence is quite significant.

Let some volume V be filled by granular particles having contact areas in common through which loads are transmitted. On the free portions of the grain surfaces loading is equal to zero. The equation of motion for the mean field will be sought in the form of an asymptotic equation, where the small parameter $\varepsilon$ will represent the ratio of the grain radius $r_0$ to the mean linear dimensions of the region V, i.e., $\varepsilon = r_0 V^{-1/3}$.

It has been shown in a number of studies [3, 4] that the displacement field differs little from the mean field in absolute value (displacement fluctuations contain a factor of the order of magnitude of $\varepsilon$). However, with regard to intensity fluctuation deformations and stresses are comparable in size to the mean stresses and deformations. Considering this fact, we write the displacement field in some volume V containing a large number of particles, yet small enough that its linear dimensions are much less than the wavelength of the propagating wave, in the form

$$U_i(x, t) = u_i(x, t) + \varepsilon v_i(x/\varepsilon, t). \tag{1}$$

In Eq. (1) the displacement field $U_i(x, t)$ depends on three spatial coordinates, denoted by the single symbol x, and on time t. The first term depends on the "slow" variable x and represents the large scale field, while the second depends on the "fast" variable $x/\varepsilon$ and is the fluctuation component. Assuming that $u_i$ and $v_i$ are of the same order of smallness, then $u_i(x, t) \gg \varepsilon v_i(x/\varepsilon, t)$. Consequently, we have the obvious relationships

$$\frac{\partial U_i}{\partial x_k} = \frac{\partial u_i}{\partial x_k} + \frac{\partial v_i}{\partial y_k}, \quad e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right); \tag{2}$$

$$\tilde{e}_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_k} + \frac{\partial v_k}{\partial y_i}\right), \quad E_{ik} = \bar{e}_{ik} + \tilde{e}_{ik}. \tag{3}$$

In Eqs. (2), (3) $y_j = x_j/\varepsilon$, the total deformation tensor $E_{ik}$ is composed of the mean tensor $\bar{e}_{ik}$ and the fluctuation tensor $\tilde{e}_{ik}$. It is evident from Eqs. (2), (3) that the deformations $\tilde{e}_{ik}$ are of the same order as the deformations $\bar{e}_{ik}$, although the mean value $\langle\tilde{e}_{ik}\rangle = 0$.

The stress field in the skeleton $\Sigma_{ik} = \lambda(\bar{\theta} + \tilde{\theta})\delta_{ik} + 2\mu(\bar{e}_{ik} + \tilde{e}_{ik})$ also consists of large and fine scale fields. We will write the difference

$$\operatorname{div} Q - E = \frac{\partial \Sigma_{ik}}{\partial x_k} U_i, \quad Q_i = \Sigma_{ik} U_k, \tag{4}$$

where E is the density of the potential energy of deformation, $\int\limits_V E dV = \int\limits_S P_i u_i dS$; $u_i$ is the displacement vector; $P_i$ is the loading vector; V is the volume of the region; and S is the surface bounding the volume V. Under conditions of quasistatic equilibrium (which occurs in deformation of a volume V, the linear dimensions of which are small in comparison to the wavelength) the divergence of the Umov–Poynting vector $E = \operatorname{div} Q$ is exactly equal to the elastic energy density of the body, so from Eq. (4) we have an equilibrium equation in the form

$$\partial \Sigma_{ik} / \partial x_k = 0. \tag{5}$$

However for the mean field $\sigma_{ik}$ we have no equation analogous to Eq. (5), since

$$\left(\frac{\partial \bar{\sigma}_{ik}}{\partial x_k} + \frac{\partial \tilde{\sigma}_{ik}}{\partial x_k}\right)(\bar{u}_i + \tilde{u}_i) = \frac{\partial \bar{\sigma}_{ik}}{\partial x_k}\bar{u}_i + \tilde{f}_i \tilde{u}_i. \tag{6}$$

It was considered in Eq. (6) that the values of the products of the mean field by the fluctuation field are equal to zero (on the average). However, this cannot be said of the product of the fluctuation force $\tilde{f}_i = \partial\tilde{\sigma}_{ik}/\partial x_k$ times the fluctuation displacement $\tilde{u}_i$. This quantity is of the order of magnitude of the fluctuation energy. Substituting Eq. (1) in Eq. (4), we obtain

$$\frac{\partial \Sigma_{ik}}{\partial x_k} = (u_i + \varepsilon v_i)\left\{\frac{1}{\varepsilon}\left(\lambda \frac{\partial \tilde{\theta}}{\partial y_k}\delta_{ik} + 2\mu \frac{\partial \tilde{e}_{ik}}{\partial y_k}\right) + \frac{\partial \bar{\sigma}_{ik}}{\partial x_k} + \frac{\partial \lambda}{\partial x_k}(\bar{\theta} + \tilde{\theta})\delta_{ik}\right.$$
$$\left. + 2\frac{\partial \mu}{\partial x_k}(\bar{e}_{ik} + \tilde{e}_{ik})\right\} = u_i \frac{\partial \bar{\sigma}_{ik}}{\partial x_k} + \lambda v_i \operatorname{div} v_i + \mu v_i \frac{\partial}{\partial y_k}\left(\frac{\partial v_i}{\partial y_k} + \frac{\partial v_k}{\partial y_i}\right) + (u_i + \varepsilon v_i)\left\{\frac{\partial \lambda}{\partial x_k}(\bar{\theta} + \tilde{\theta})\delta_{ik} + 2\frac{\partial \mu}{\partial x_k}(\bar{e}_{ik} + \tilde{e}_{ik})\right\}. \tag{7}$$

In the second and third terms of Eq. (7) we have expressions of the type

$$v_1 \frac{\partial^2 v_1}{\partial y_1^2} = \frac{1}{2} \frac{\partial^2 v_1^2}{\partial y_1^2} - \left(\frac{\partial v_1}{\partial y_1}\right)^2.$$

Consequently,

$$(\lambda + 2\mu)\left(\frac{1}{2} \frac{\partial^2 v_1^2}{\partial y_1^2} - \tilde{e}_{11}^2\right) = -\tilde{\sigma}_{11}\tilde{e}_{11} + \frac{\lambda + 2\mu}{2} \frac{\partial^2 v_1^2}{\partial y_1^2},$$

and similarly $(\lambda + 2\mu)\left(v_2 \frac{\partial^2 v_2}{\partial y_2^2} + v_3 \frac{\partial^2 v_3}{\partial y_3^2}\right) = -\tilde{\sigma}_{22}\tilde{e}_{22} - \tilde{\sigma}_{33}\tilde{e}_{33} - \frac{\lambda + 2\mu}{2}\left(\frac{\partial^2 v_2^2}{\partial y_2^2} + \frac{\partial^2 v_3^2}{\partial y_3^2}\right)$. We will attempt to average quantities of the type $\partial^2 v^2/\partial y^2$ at distances $l$ much larger than the grain size but at the same time small in comparison to the linear characteristic dimensions of the nonsteady state process (for example, cT, where T is the time the signal acts at the source). Then

$$\left\langle \frac{\partial^2 v^2}{\partial y^2} \right\rangle = \frac{1}{l}\left\{\frac{\partial v^2}{\partial y}\bigg|_{x+l} - \frac{\partial v^2}{\partial y}\bigg|_{x}\right\}.$$

We can perform a second averaging

$$\left\langle \frac{\partial^2 v^2}{\partial y^2} \right\rangle = \frac{1}{l^2}\left\{v^2\,|_{x+l} - v^2\,|_{x}\right\}.$$

Since $v^2$ is a random function at points of the medium, it is equally probable to obtain a difference $v_{x+l}^2 - v_x^2$ positive or negative in sign. Thus, obviously, the mean value of this difference is equal to zero. We obtain the contribution from this type of expression in the form of the fluctuation component of the compression energy, i.e., $\tilde{\sigma}_{ii}\tilde{e}_{ii}$. In detailed notation the three terms of Eq. (7) have the form

$$u_1 \frac{\partial \sigma_{1k}}{\partial x_k} + u_2 \frac{\partial \sigma_{2k}}{\partial x_k} + u_3 \frac{\partial \sigma_{3k}}{\partial x_k} + \lambda\left\{\left(v_1 \frac{\partial}{\partial y_1} + v_2 \frac{\partial}{\partial y_2} + v_3 \frac{\partial}{\partial y_3}\right)\operatorname{div} v\right\}$$
$$+ \mu\left\{2\left(v_1 \frac{\partial^2 v_1}{\partial y_1^2} + v_2 \frac{\partial^2 v_2}{\partial y_2^2} + v_3 \frac{\partial^2 v_3}{\partial y_3^2}\right) + v_1 \frac{\partial}{\partial y_2}\left(\frac{\partial v_2}{\partial y_1} + \frac{\partial v_1}{\partial y_2}\right)\right.$$
$$\left. + v_1 \frac{\partial}{\partial y_3}\left(\frac{\partial v_1}{\partial y_3} + \frac{\partial v_3}{\partial y_1}\right) + v_2 \frac{\partial}{\partial y_1}\left(\frac{\partial v_2}{\partial y_1} + \frac{\partial v_1}{\partial y_2}\right) + v_2 \frac{\partial}{\partial y_3}\left(\frac{\partial v_2}{\partial y_3} + \frac{\partial v_3}{\partial y_2}\right) + v_3 \frac{\partial}{\partial y_1}\left(\frac{\partial v_3}{\partial y_1} + \frac{\partial v_1}{\partial y_3}\right) + v_3 \frac{\partial}{\partial y_2}\left(\frac{\partial v_2}{\partial y_3} + \frac{\partial v_3}{\partial y_2}\right)\right\}.$$

Expressions of the type

$$\mu\left\{v_1 \frac{\partial^2 v_1}{\partial y_2^2} + v_2 \frac{\partial^2 v_2}{\partial y_1^2}\right\}$$

transform to the form

$$\mu\left\{-\left(\frac{\partial v_1}{\partial y_2}\right)^2 - \left(\frac{\partial v_2}{\partial y_1}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 v_2^2}{\partial y_1^2} + \frac{\partial^2 v_1^2}{\partial y_2^2}\right)\right\}.$$

Adding and subtracting the quantity $2\frac{\partial v_1}{\partial y_2}\frac{\partial v_2}{\partial y_1}$, we have

$$\mu\left\{-\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right)^2 + 2\frac{\partial v_1}{\partial y_2}\frac{\partial v_2}{\partial y_1} + \frac{1}{2}\left(\frac{\partial^2 v_1^2}{\partial y_2^2} + \frac{\partial^2 v_2^2}{\partial y_1^2}\right)\right\}.$$

For mean values, considering that $\left\langle \frac{\partial v_1}{\partial y_2}\frac{\partial v_2}{\partial y_1} \right\rangle = 0$, $\left\langle \frac{\partial^2 v_2^2}{\partial y_1^2} \right\rangle = \left\langle \frac{\partial^2 v_1^2}{\partial y_2^2} \right\rangle = 0$, we obtain a contribution in the form $2\mu\tilde{e}_{12}^2 = 2\tilde{\sigma}_{12}\tilde{e}_{12}$, i.e., the shear components of the fluctuation energy. Since the mean values of the cross products are equal to zero when i, k differ from each other, after averaging the first three terms of Eq. (7) take on the form

$$\frac{\partial \bar{\sigma}_{ik}}{\partial x_k}\bar{u}_i - \tilde{E}_i, \quad \tilde{E}_i = \tilde{\sigma}_{ik}\tilde{e}_{ik}.$$

It remains to clarify the meaning of the term

$$(u_i + \varepsilon v_i)\left\{\frac{\partial\lambda}{\partial x_k}(\bar{\theta}+\tilde{\theta})\,\delta_{ik} + 2\frac{\partial\mu}{\partial x_k}(\bar{e}_{ik}+\tilde{e}_{ik})\right\},$$

or

$$U_i\left\{\frac{\partial\lambda}{\partial x_k}\Theta\delta_{ik} + 2\frac{\partial\mu}{\partial x_k}E_{ik}\right\}. \tag{8}$$

First of all we note that $\partial\lambda/\partial x_k$, $\partial\mu/\partial x_k$ are concentrated at $r = r_0$, the grain boundaries, or more precisely, on the portions of the particles free of loading, i.e., on the free surface of the pores, where they take on infinite values. In the internal region of a grain and on the contact surface $\partial\lambda/\partial x_k = \partial\mu/\partial x_k = 0$. The sign of these products on the loadfree grain surface is determined by the cosine of the angle formed between the axis and the direction of the radius vector from the center of the grain to the point of the free surface in question. Therefore, for points of the free surface $\partial\lambda/\partial x_k = \partial\lambda n_k/\partial r = \delta(r)n_k$, where $n_k = \cos(n, x_k)$. We average Eq. (8) over grain volume:

$$\frac{1}{V_0}\int\limits_{V_0}\left[\frac{\partial\lambda}{\partial x_k}\Theta\delta_{ik} + 2\frac{\partial\mu}{\partial x_k}E_{ik}\right]U_i dV = \frac{1}{V_0}\int\limits_{S*}U_i(\lambda\Theta\delta_{ik} + 2\mu E_{ik})\,n_k dS = \frac{1}{V_0}\int\limits_{S*}U_i\Sigma_{ik}n_k dS.$$

Here S* is the portion of the grain surface free of loading. In view of the obvious relationship between stresses and loads we have

$$\frac{1}{V_0}\int\limits_{S*}\Sigma_{ik}n_k U_i dS = \frac{1}{V_0}\int\limits_{S*}P_i U_i dS = 0. \tag{9}$$

The integral of Eq. (9) is the deformation energy, which is concentrated on the free surface of the grain. That latter quantity is equal to zero in view of the obvious relationship for an empty skeleton, $P_i|_{S*} = 0$. Thus, the mean value of Eq. (8) is equal to zero. Then averaging of the expression $U_i\partial\Sigma_{ik}/\partial x_k$ leads to equilibrium equations

$$\bar{u}_i\partial\bar{\sigma}_{ik}/\partial x_k = \tilde{E}_i, \tag{10}$$

where $\tilde{E}_i$ is the portion of the fluctuation energy produced by products of stresses and deformations with fixed subscript i. Since the fluctuation energy is a function of the mean energy E, we write it in the form

$$\tilde{E} = A + \alpha\bar{E} + \beta E^2 + \ldots \tag{11}$$

Considering the absence of a fluctuation field upon disappearance of the mean field, we must take A = 0. If we limit ourselves to the first term in the expansion of Eq. (11), then we obtain the expression $\tilde{E} = \alpha\bar{E}$, where $\alpha$ is a constant dependent on the structure of the pore space and equal to zero for a continuous medium.

Equilibrium equation (10) takes on the form $\partial\bar{\sigma}_{ik}/\partial x_k = \alpha\bar{E}_i/\bar{u}_i$. The equation of motion is related to consideration of inertial forces, and according to Eq. (1) inertial forces are almost totally (to the accuracy of $\varepsilon$) determined by the acceleration of the mean field. Therefore the equations of motion can be represented in the form

$$\partial\bar{\sigma}_{ik}/\partial x_k - \alpha\bar{E}_i/\bar{u}_i = \rho\ddot{\bar{u}}_i, \tag{12}$$

where $\rho$ is the mean density of the structure. In the future we will omit the bar symbol above the average field. In order to complete system (12), it is necessary to relate the mean stresses $\sigma_{ik}$ to the mean deformations. This is a unique problem and can be solved exactly for an individual grain by the methods of boundary integral equations [5].

In the presence of an average longitudinal plane wave all but one of the deformations (of the mean field) are equal to zero, so that the mean energy $E = \sigma\cdot e = (\lambda + 2\mu)(\partial u/\partial x)^2$. In this case nonlinear equation (12) has the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\alpha}{u}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{13}$$

and at $\alpha = 0$ transforms to the conventional wave equation for a continuous medium with $c^2 = (\lambda + 2\mu)/\rho_0(1 - f)$, where f is the porosity of the skeleton, $\rho_0$, $\mu$ is the density of the grain material, and $\lambda$, $\mu$ are the mean values of the Lamé coefficients on the structure. The constants c and $\alpha$ are completely defined by the structure of the pore space, i.e., the porosity f, the product $\eta = \sigma_0 r_0/3$ of the specific surface times the mean grain radius, and the average number of contacts n.

If for Eq. (13) we pose the problem of wave propagation for instantaneous application of an isolated load, then the problem contains no characteristic times or distances.

The dimensionless deformation will depend only on the single dimensionless variable $\xi = x/ct$. Such solutions are termed self-similar. We will seek a solution of the problem of integration of Eq. (13) with a boundary condition of the form

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \begin{cases} 0, & t<0, \\ e_0(0), & t\geqslant 0. \end{cases}$$

We set $u = ct[\bar{u}(\xi) + c_0]$, where $u(\xi)$ is the dimensionless function, and $c_0$ is a dimensionless constant. Then

$$\frac{\partial u}{\partial x} = \bar{u}'_\xi, \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{ct}\bar{u}''_{\xi\xi},$$

$$\frac{\partial u}{\partial t} = c\left[\bar{u} - \xi\bar{u}'_\xi + c_0\right], \quad \frac{\partial^2 u}{\partial t^2} = \xi^2\bar{u}''_{\xi\xi}\frac{1}{ct}.$$

One of the variables is thus eliminated, and in the case of self-similar motions Eq. (13) transforms to an ordinary differential equation (the bar above the dimensionless functions is omitted)

$$u''(u + c_0)(1 - \xi^2) = \alpha u'^2.$$

The boundary condition for Eq. (14) is formulated on the line $\xi = 0$, i.e., at $x = 0$, where the deformations $u'(0)$ are specified. Equation (14) admits an exact solution, since upon the substitution $u'/u + c_0 = v$ it transforms to an equation with separable variables. Instead of Eq. (14) we have a first-order equation

$$v' + v^2(1 - \alpha/(1 - \xi^2)) = 0,$$

which can be solved in the form

$$v = \frac{1}{\xi + D + \frac{\alpha}{2}\ln\frac{1-\xi}{1+\xi}},$$

where D is an arbitrary constant. It is obvious that as $\xi \to 0$, i.e., on the boundary $x = 0$, $u' \to (u + c_0)$, as in conventional elasticity, where $u = A(\xi - 1)$, and the deformation $u'$ is constant. This requirement defines the value $D = -1$. Therefore

$$v = \frac{u'}{u + c_0} = \frac{1}{\xi - 1 + \frac{\alpha}{2}\ln\frac{1-\xi}{1+\xi}}. \tag{15}$$

We now choose the constant $c_0$ such that the displacements on the front $\xi = 1$ are equal to zero. This requirement determines the final form of the solution

$$u = A\left\{\exp\int_0^\xi v(x, \alpha)\,dx - \exp\int_0^1 v(x, \alpha)\,dx\right\}, \tag{16}$$

where

$$A = \frac{-e_0(0)}{1 - \exp\int_0^1 v(x, \alpha)\,dx}.$$

In Eq. (16) the second term is constant, insuring equality to zero of the displacements on the wave front $\xi = 1$ and ahead of it. As $\alpha \to 0$ solution (16) transforms to a conventional elastic self-similar solution, i.e., constant deformation behind the front of the wave, with a discontinuity at the front itself. At $\alpha \neq 0$ the deformations behind the wavefront $\xi = 1$ are no longer constant, but are functions of the variable $\xi$. It is also significant that the deformation (and particle velocity) at the front $\xi = 1$ are equal to zero in view of the infinite value of the logarithm in Eq. (15). Consequently, in the model considered strong discontinuities cannot exist. Deformations set in smoothly, and P and S waves can only be weak discontinuity waves. It is interesting that the energy of the wave propagates not only at velocities close to c, but also at any other velocities less than c. This fact greatly complicates interpretation of waves in microinhomogeneous media, since the energy of the initial disturbances is extremely small in comparison to the energy of slower waves. Concrete calculations of displacements and deformations as a function of $\xi$ for various values of $\alpha$ are shown in Fig. 1a, b, where $\alpha = 0.01$ and 0.11, respectively, the solid lines are dimensionless displacement u/ct vs $\xi$, and the dashes are the dependence of deformation on $\xi$.

Equation (13) with c = 1 has a solution in the form of waves propagating in one direction at a still unknown velocity, i.e., $u = f(t - x/a)$. Substituting this expression in Eq. (13), we obtain the ordinary equation

$$\left(\frac{1}{a^2} - 1\right)\frac{f''}{f'} = \frac{\alpha}{a^2}\frac{f'}{f},$$

whence

$$u = C\,(t - x/a)^{\frac{1}{(1-\alpha)/(1-a^2)}}, \tag{17}$$

where C is an arbitrary constant. If in solution (17) we take the exponent $1/[(1 - \alpha)/(1 - a^2)] = n$, then the wave velocity $a$ is a function

$$a(n, \alpha) = \sqrt{1 - n\alpha/(1 - n)}.$$

Thus, with a linear increase in deformation (quadratic increase in displacement) n = 2 and correspondingly $a = \sqrt{1 - 2\alpha} \approx 1 - \alpha$. Thus, in the absence of a discontinuity in deformation the wave velocity is significantly lower. With further increase in n the velocity increases, tending to the limit $a = \sqrt{1 - \alpha}$, which is always less than in the case of a suddenly applied load. If $n = 1/(1 - \alpha)$, i.e., the deformations increase very slowly [by a law $e(0, t) = t^{\alpha/(1-\alpha)}$], then $a = 0$, and in general, waves will not propagate. This result differs greatly from that of pure wave processes, for which waves propagate for any law of loading change with time. Thus, depending on the type of load applied, the wave velocities decrease significantly. Therefore, in experiments an entire zone of unstable oscillation reception can develop in the range from $c(1 - \alpha)$ to c.

For a wave process we have the relationship $u = f(t - x)$, and consequently $u_t + u_x = 0$. This is obviously not the case for Eq. (13), however the hypothesis that $u_t + u_x = O(\alpha)$ develops. In the characteristic variables $\xi = t - x$, $\eta = t + x$ the equation of motion has the form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{\alpha}{4u}\left(\frac{\partial u}{\partial\eta} - \frac{\partial u}{\partial\xi}\right)^2 = 0. \tag{18}$$

Taking $\partial u/\partial\eta = v$, we note that for the function $v(\xi, \eta)$ Eq. (18) is the Riccati equation

$$\frac{\partial v}{\partial\xi} + \frac{\alpha}{4u}v^2 - \frac{\alpha}{2u}\frac{\partial u}{\partial\xi}v + \frac{\alpha}{4u}\left(\frac{\partial u}{\partial\xi}\right)^2 = 0.$$

If the hypothesis that $\partial u/\partial\eta = O(\alpha)$ is valid, then the term $\alpha v^2/4u = O(\alpha^3)$, and may be neglected. In this case the solution is obvious:

$$\frac{\partial u}{\partial\eta} = -\frac{\alpha}{4}u^\alpha\int u^{-(1+\alpha)}\left(\frac{\partial u}{\partial\xi}\right)^2 d\xi. \tag{19}$$

If we neglect terms of the order of $\alpha^2$, then Eq. (19) can be replaced by

$$\frac{\partial u}{\partial\eta} = -\frac{\alpha}{4}\int u^{-1}\left(\frac{\partial u}{\partial\xi}\right)^2 d\xi. \tag{20}$$

Thus it proves to be the case that $\partial u/\partial\eta$ is of the order of magnitude of $\alpha$. This justifies the hypothesis made above. Now it can easily be proved that the wave operator applied to the combination $u_x + u_t$ vanishes to the accuracy of $O(\alpha^2)$, i.e., $\Box(u_x + u_t) = O(\alpha^2)$. Therefore, the linearized equation of motion is a third-order equation

$$\Box(u_x + u_t) = 0. \tag{21}$$

The general solution of Eq. (21) is:

$$u = \alpha[f_1(\xi) + f_2(\eta)] + f_3(\xi).$$

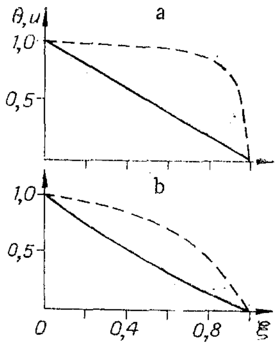


Fig. 1

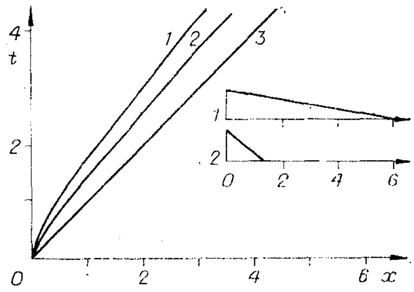


Fig. 2

where $f_1$, $f_2$, $f_3$ are arbitrary functions. The presence of three arbitrary functions permits solution of the important question of the configuration of the weak discontinuity wave front $x = x_0(t)$. In fact, on the boundary $x = 0$ the deformations $\partial u(0, t)/\partial x = h(t)$ are specified on the weak discontinuity front $u_x = u_t = 0$, and moreover, on the boundary $x = 0$ from Eq. (20) we have one more condition $u_x + u_t|_{x=0} = -\frac{\alpha}{2}M(t)$, where $M(t) \approx \int_0^t \frac{\dot u^2(0, t)}{u(0, t)}dt$. The four conditions enumerated completely determine the three unknown functions and the front configuration $x = x_0(t)$. Setting $x_0(t) = t\zeta(t)$, we write

$$-\alpha\left[f_2'(t) + \dot M(t)/4\right] t\frac{2\zeta}{1-\zeta} = h(t) + \alpha M(t)/4,$$
$$f_2[t(1+\zeta)/(1-\zeta)] - f_2(t) = M(t)/4. \tag{22}$$

Eliminating the function $f_2$ from Eq. (22) we obtain the functional equation of the front, i.e., the function $\zeta(t)$. Concrete calculation of the front for a sinusoidal (with reference to deformations) signal products a result in which initially the front moves with a velocity $c(1-\alpha)$, which to the accuracy of $\alpha^2$ coincides with the exact solution for a linearly increasing load. Then the front velocity increases, although under all circumstances $x_0(t)/t < c$.

Upon abrupt application of loading a region exists in which linearization of the motion is impossible. In this case the motion must be linearized in a region bounded by the line $x = 0$ and some weak discontinuity wave $x = x_0(t)$, subject to determination. In the region included between the characteristic $x = ct$ and the front $x = x_0(t)$ the motion will obviously be self-similar [Fig. 2, $\alpha = 0.1$, lines 1, 2 correspond to $x = x_0(t)$, line 3, $x = ct$].

In the region bounded by the lines $x = x_0(t)$ and $x = 0$ let the motion be described by an equation $u'' = \ddot u/c^2$, so that

$$u(x, t) = f_1(t - x/c) + f_2(t + x/c),$$

where $f_1$, $f_2$ are arbitrary functions. Correspondingly,

$$c\frac{\partial u}{\partial x} = f_2'(t + x/c) - f_1'(t - x/c), \quad \frac{\partial u}{\partial t} = f_2'(t + x/c) + f_1'(t - x/c).$$

On the second weak discontinuity front (in view of the continuity of velocities and deformations)

$$c[\bar u(\xi) - \xi\bar u'(\xi)] = f_1'[t - x_0(t)/c] + f_2'[t + x_0(t)/c]; \tag{23}$$

$$c\bar u'(\xi) = f_2'[t + x_0(t)/c] - f_1'[t - x_0(t)/c]. \tag{24}$$

Moreover, on the boundary $x = 0$ the deformation $e_0(t)$ is specified, i.e.,

$$c\partial u(0, t)/\partial x = f_2'(t) - f_1'(t),$$

or

$$ce_0[t + x_0(t)/c] = f_2'[t + x_0(t)/c] - f_1'[t + x_0(t)/c]. \tag{25}$$

Equations (23)-(25) are sufficient for determination of the arbitrary functions $f_1$, $f_2$ and the equation of the second front $x = x_0(t)$. Adding and subtracting Eqs. (23), (24) with consideration of Eq. (25), we find

$$\bar{u}(\xi) + (1 - \xi)\bar{u}'(\xi) = 2e_0[t + x_0(t)/c] + 2f_1'[t + x_0(t)/c]/c,$$
$$\bar{u}(\xi) - (1 + \xi)\bar{u}'(\xi) = 2f_1'[t - x_0(t)/c]/c. \tag{26}$$

Eliminating from Eq. (26) the unknown function $f_1'$, we obtain a functional equation for the line $x = x_0(t)$. To do this we set $t(1 + \xi) = z$. Then Eq. (26) takes on the form

$$\bar{u}(z/t - 1) + (2 - z/t)\bar{u}'(z/t - 1) = 2e_0(z) + 2f'(z)/c,$$
$$\bar{u}(z/t - 1) - z\bar{u}'(z/t - 1)/t = 2f_1'[z(1 - \xi)/(1 + \xi)]/c.$$

Replacing z in the second expression by the quantity $z(1 + \xi)/(1 - \xi)$ and subtracting the second equation from the first, we obtain the functional equation of the second front, relating t and $\xi$:

$$\bar{u}(\xi) - \bar{u}[(1 + \xi)^2/(1 - \xi)] + (1 - \xi)\bar{u}'(\xi) + (1 + \xi)^2\bar{u}'[(1 + \xi)^2/(1 - \xi) - 1]/(1 - \xi) = 2e_0[t(1 + \xi)]. \tag{27}$$

Figure 2 shows calculations of the kinematics of the boundary of the second weak discontinuity, which in nonlinear mechanics is usually termed the unloading wave.

At $t = 0$ Eq. (27) is satisfied by the substitution $\xi = 0$. Consequently, the second front begins motion with zero velocity. It is evident from calculations that with passage of time the velocity of the second front approaches c, so that in the far zone hodographs of the onsets of the two fronts will be parallel straight lines. The linearized problem immediately shows attenuation of a plane wave with distance, because the wave equation is complemented not by the boundary problem of conventional elasticity, but a condition close to mixed, wherein one of the boundaries approaches the characteristic. On this boundary the displacements and deformations vanish, which determines the attenuation of the plane wave. In experiment the second front must have much higher intensity than the first, since the particle velocities and deformations at $x = ct$ are equal to zero, and the recording device can detect only second derivatives of the displacements. On the second front the displacements and deformations are nonzero.

Thus, the model considered shows that measurement of acoustical wave velocity in a microinhomogeneous medium is not at all a trivial problem. Unstable recording of wave velocities in the range from $c(1 - \alpha)$ to c (according to the data of [5] the quantity $\alpha$ for highly porous bodies may reach 0.1-0.15 or more) can lead to the appearance of negative Poisson coefficients, even when static measurements do not confirm this. The dependence of wave velocity on the form of loading applied explains the paradoxical fact that the wave velocity in such media not only depends on the medium itself, but is in some sense a function of the experimental equipment used.

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